EXTENSIONS OF THE MEASURABLE CHOICE THEOREM BY MEANS OF FORCING

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ABSTRACT

Using the method of forcing of set theory, we prove the following two theorems on the existence of measurable choice functions: Let T be the closed unit interval [0,1] and let m be the usual Lebesgue measure defined on the Borel subsets of T. Theorem 1. Let $S \subset T \times T$ be a Borel set such that for all $t \in T$, $S_t \overset{def}{=} \{x \mid (t,x) \in S\}$ is countable and non-empty. Then there exists a countable series of Lebesgue-measurable functions $f_n : T \to T$ such that $S_t = \{f_n(t) \mid n \in \omega\}$ for all $t \in T$. Theorem 2. Let $W \subset [0,1] \times [0,1]$ be a Borel set such that for each $x \in [0,1]$, $W_x = \{y \mid (x,y) \in W\}$ is uncountable. Then there exists a function $h: [0,1] \times [0,1] \to W$ with the following properties: (a) for each $x \in [0,1]$, the function $h(x, \cdot)$ is one-one and onto W_x and is Borel measurable; (b) for each y, $h(\cdot, y)$ is Lebesgue measurable; (c) the function h is Lebesgue measurable.

Let $A \subset [0,1] \times [0,1]$ be a Borel set whose projection on the x-axis consists of all points in $[0,1]$. Under these circumstances, Von Neumann [9] proved the existence of a Lebesgue measurable function F, defined on $[0,1]$, such that $F(x) \in A_x \overset{\text{def}}{=} \{y \mid (x,y) \in A\}$ for all $x \in [0,1]$.

Through the use of forcing, we extend the foregoing result in several directions. Theorem 1 deals with the case where $\{y \mid (x,y) \in A\}$ is countable for every $x \in [0,1]$. We prove the existence of a countable series of Lebesgue measurable functions $\{f_i\}$ such that $\{f_i(x)\} = \{y \mid (x,y) \in A\}$ for every $x \in [0,1]$. In Theorem 2, $\{y \mid (x,y) \in A\}$ is uncountable for all $x \in [0,1]$. We then produce an analog of Theorem 1. In another paper [10], we consider a situation arising in markets with a continuum of traders; there an affirmative answer is given to a problem formulated by R. J. Aumann and G. Debreu [1] using the same techniques as those presented here.

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For an understanding of the proofs in this articles, the reader should be familiar with the theorems and concepts relating to forcing in set theory. An acquaintance with the ideas developed in Solovay's paper [8, Chapter II] would also be helpful.

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1.

Let T be the closed unit interval $[0,1]$ and let m be the usual Lebesgue measure defined on the Borel subsets of T. We denote by $\mathscr{R}$ the set of real numbers.

THEOREM 1. *Let $S \subset T \times \mathscr{R}$ be a Borel set such that for all $t \in T$, $S_t \overset{def}{=} \{x \mid (t,x) \in S\}$ is countable and non-empty. Then there exists a countable series of Lebesgue-measurable functions $f_n : T \to \mathscr{R}$ such that $S_t = \{f_n(t) \mid n \in \omega\}$ for all $t \in T$.*

PROOF. In accordance with [8, Chapter II, Section 1], there exist $\prod_1^1$ predicates $A_2(x_1,x_2)$, $A_3(x_1,x_2)$ which can be used to code every Borel subset of the real line, i.e., for every Borel subset $E \subset \mathscr{R}$ there exists a fixed parameter $\eta \in \omega^\omega$ such that for all $x \in \mathscr{R}$, $x \in E \leftrightarrow A_2(\eta,x)$ and $x \notin E \leftrightarrow A_3(\eta,x)$ Similarly there are $\prod_1^1$ predicates $Q_2(x_1,x_2,x_3)$, $Q_3(x_1,x_2,x_3)$ which may be used to code every Borel subset $D \subset \mathscr{R} \times \mathscr{R}$: for some $\nu \in \omega^\omega$, $(x,y) \in D \leftrightarrow Q_2(\nu,x,y)$, $(x,y) \notin D \leftrightarrow Q_3(\nu,x,y)$. Let P be an arithmetical formula which provably defines a $1-1$ correspondence p between $[0,1]$ and ω^ω. We shall often identify $a \in [0,1]$ with $p(a)$ without specific mention. This convention will allow us to code all Borel subsets of ω^ω by means of the same predicates $A_2(x_1,x_2)$, $A_3(x_1,x_2)$. For any $\lambda, \mu \in \omega^\omega$ which code Borel subsets of $\mathscr{R}$ and $\mathscr{R} \times \mathscr{R}$ respectively, let $B_{1,\lambda} \subset \mathscr{R}$, $B_{2,\mu} \subset \mathscr{R} \times \mathscr{R}$ be the Borel sets coded by λ and μ. Further, let $\xi_1 \in \omega^\omega$ be some parameter which codes S, i.e., $S = B_{2,\xi_1}$. We now choose a countable transitive model M of ZFC such that $\xi_1 \in M$ and such that every set in M is constructible from ξ_1. The existence of a model satisfying these conditions cannot, of course, be proven within Zermelo-Fraenkel set theory (see [2, p. 78]). However, an analysis of the present proof will show that M need only fulfill some finite subset of $ZFC + \xi_1$-constructibility in order for the proof to work. The existence of transitive models of any finite subset of $ZFC + \xi_1$-constructibility can be demonstrated using only the axioms of ZFC [2, p. 82]. Thus, our proof can, in fact, be carried out entirely within ZFC. Nevertheless, to simplify the exposition of the proof, we shall continue to assume that M fulfills all of the axioms of ZFC.

For any $\mu \in \omega^\omega$ which codes a Borel subset of $\mathscr{R} \times \mathscr{R}$ and any transitive model

N of ZFC containing μ, let $B^N_{2,\mu} = B_{2,\mu} \cap N$. By virtue of the absoluteness of $\prod^1_1$ statements [7, pp. 137–138], $B^N_{2,\mu} = \{(x,y) \mid x, y \in N \,\&\, N \Vdash Q_2(\mu, x, y)\}$ since $Q_2(\mu, x, y)$ is $\prod^1_1$. $B^N_{2,\mu}$ is therefore definable in N and thus $B^N_{2,\mu} \in N$. $B^N_{1,\lambda}$ is defined analogously.

Let $t \in \mathscr{R}$ be random with respect to M, i.e., for every μ in M which codes a Borel subset of $\mathscr{R}$, if $m(B_{1,\mu}) = 0$, then $t \notin B_{1,\mu}$. Since M is countable, almost every $t \in \mathscr{R}$ in the real world is random over M. Using an adaption of the label space procedure [2, Chapter IV], we obtain a uniquely defined model $M[t]$ of ZFC containing t, such that $M \subset M(t)$. In $M(t)$, every element is constructible from ξ_1 and t.

Let $\mathscr{L}$ be the language of set theory enriched by names for each of the members of M. For any $\lambda, \mu \in M$, where λ and μ code Borel subsets of $\mathscr{R}$ and $\mathscr{R} \times \mathscr{R}$ respectively, let $\bar{B}_{1,\lambda}$ and $\bar{B}_{2,\mu}$ be terms in $\mathscr{L}$ denoting the sets $B^M_{1,\lambda}$ and $B^M_{2,\mu}$ in M, i.e., the terms defined as $\{x \mid A_2(\lambda, x)\}$, $\{(x,y) \mid Q_2(\mu, x, y)\}$. We expand $\mathscr{L}$ to $\mathscr{L}'$ by adding a new symbol, $\bar{t}$. $\bar{t}$ will denote t in $M[t]$. Every member of $M[t]$ is then uniquely determined by some term in $\mathscr{L}'$ [2, Chapter IV]. In particular, for every $\lambda \in M[t]$ that codes a Borel subset of $\mathscr{R} \times \mathscr{R}$, $B^{M[t]}_{2,\lambda}$ is denoted by the term $\bar{B}_{2,\lambda}$ of $\mathscr{L}'$.

Since the sets in $M[t]$ are constructible from ξ_1 and t, the elements in ω^ω belonging to $M[t]$ are well ordered by the ordinals and sentences through which they are constructed. Thus, there exists a fixed formula $\Theta(x, y)$ in $\mathscr{L}'$ which, for each random t, defines a well ordering over the members of ω^ω appearing in $M[t]$, when interpreted in $M[t]$.

For every random t, the statement "$\bar{B}_{2,\xi_1}$ is Borel" holds true in $M[t]$. Thus $M[t] \Vdash$ "$\{y \mid (t, y) \in \bar{B}_{2,\xi_1}\}$ is Borel", since $\{y \mid (t,y) \in B^{M[t]}_{2,\xi_1}\}$ is the cross-section of a Borel set. Hence there exists some $\eta \in \omega^\omega$ in $M[t]$ such that

$$(1) \qquad \text{“}\forall y (y \in \bar{B}_{1,\eta} \leftrightarrow (t,y) \in \bar{B}_{2,\xi_1})\text{”}$$

holds true in $M[t]$. However, (1) is equivalent to

$$\forall y ([A_2(\eta, y) \,\&\, Q_2(\xi_1, t, y)] \vee [A_3(\eta, y) \,\&\, Q_3(\xi_1, t, y)])$$

which is $\prod^1_1$ and therefore absolute. Thus (1) also holds true in the real world. By hypothesis, the set $\{y \mid (t,y) \in B_{2,\xi_1}\}$ is countable for all $t \in [0,1]$. Therefore, $B_{1,\eta}$ is countable in the real world. But by Lemma 7 [8, p. 22] the statement "$\bar{B}_{1,\eta}$ is countable" is absolute. Thus

$$(2) \qquad M[t] \models \text{“}\{y \mid (t,y) \in \bar{B}_{1,\eta}\} \text{ is countable”}$$

for all random $t \in [0,1]$. Let $\eta_t \in \omega^\omega$ be the first η in $M[t]$ (under the well-ordering induced by Θ) satisfying (1).

We define v on $\omega^\omega \times \omega$ in the following manner:

$$v(\alpha,0) = \{\alpha(1), \alpha(3), \alpha(5), \cdots\}$$
$$v(\alpha,1) = \{\alpha(2), \alpha(6), \alpha(10), \cdots\}$$
$$\cdots$$
$$v(\alpha,n) = \{\alpha(2^n \cdot (2 \cdot 1 - 1)), \alpha(2^n \cdot (2 \cdot 2 - 1)), \cdots, \alpha(2^n \cdot (2 \cdot m - 1)), \cdots\}.$$
$$\cdots$$

Then for every sequence $\alpha_1, \alpha_2, \cdots$ of elements in ω^ω, there exists an $\alpha \in \omega^\omega$ such that $v(\alpha,1) = \alpha_1, v(\alpha,2) = \alpha_2, \cdots$. v is clearly definable in every transitive model M of ZFC. v is also readily seen to be recursive.

By virtue of the fact that $M[t] \models$ "$\{y \mid y \in \bar{B}_{1,\eta_t}\}$ is countable" there exists an $\alpha \in \omega^\omega$ belonging to $M[t]$ such that

$$M[t] \models \forall y[y \in \bar{B}_{1,\eta_t} \leftrightarrow \exists n(v(\alpha,n) = y)]. \tag{3}$$

Let $\alpha_t \in \omega^\omega$ be the first α in $M[t]$ (under the well ordering induced by Θ) satisfying (3). Then

$$\text{``}\forall y(y \in \bar{B}_{1,\eta_t} \leftrightarrow \exists n(v(\alpha_t,n) = y)]\text{''} \tag{4}$$

holds true in $M[t]$. As in the case of (1), (4) may also be represented as a $\prod_1^1$ relation. Due to the absoluteness of $\prod_1^1$ relations, (4) is then fulfilled in the real world.

η_t and α_t are uniquely determined for each random t. For any rational q_1 and any natural n_1, let D_{q_1,n_1} be the set of random reals defined by

$$t \in D_{q_1,n_1} \leftrightarrow t \text{ is random and } q_1 \leqq v(\alpha_t, n_1).$$

By reviewing the methods used to define α_t, we can readily verify that the relation "$q_1 \leqq v(\alpha_t, n_1)$"is expressible in $\mathscr{L}'$. That is, for every rational q_1 and every natural n_1, there exists a sentence $\Phi_0(\bar{t})$ in $\mathscr{L}'$ such that for all random t,

$$q_1 \leqq v(\alpha_t, n_1) \leftrightarrow (M[t] \models \Phi_0(t)).$$

Applying Theorem II, 2.8 of [8, p. 38]†, we obtain that D_{q_1,n_1} is Borel.

For each fixed $x \in [0,1]$, the set $S_x \overset{def}{\equiv} \{y \mid (x,y) \in B_{2,\xi_1}\}$ is countable. Let $\tilde{S}_x$ be the set of functions from the positive integers onto S_x. In accordance with the axiom of choice, there exists a function $\mathscr{F}$ defined on $[0,1]$ such that $\mathscr{F}(x) \in \tilde{S}_x$

† This theorem says essentially that for every sentence $\Phi(\bar{t})$ in $\mathscr{L}'$, the set $\{t \mid t$ is random & $M[t] \models \Phi(t)\}$ is Borel.

for each $x \in [0,1]$. We now define the function $\Omega(t,n)$, where $t \in [0,1]$ and n is a positive integer.

$$\Omega(t,n) = \begin{cases} v(\alpha_t, n) & \text{for random } t \\ \mathscr{F}(t)(n) & \text{for non-random } t. \end{cases}$$

Since $D_{q,n}$ is Borel for every rational q and every natural n, $\{t \mid \Omega(t,n_1) \geqq q_1\}$ is Lebesgue measurable for all n_1, q_1. It follows that $\Omega(t,n)$ is Lebsegue measurable for each fixed n. Through the absoluteness of (4), we have that for every fixed $x_1 \in [0,1]$, the set of values of $\Omega(x_1,n)$ coincides with A_{x_1}. Thus, if we set $f_i(x) = \Omega(x,i)$, we find that $\{f_i\}$ has all of the desired properties.

To show that the above proof can be carried out in ZFC, we make use of the fact that for any finite set $\mathscr{T}$ of the axioms of ZFC the following is a theorem of ZFC: For any $c \in \omega^\omega$, there exists a countable transitive model M such that

(1) $M \models \mathscr{T}, c \in M$;

(2) $M \models$ "Every set is constructible from c";

(3) for every real t which is random over M, there exists a uniquely defined transitive model $M[t]$ such that

(a) $M \subset M[t]$; $t \in M[t]$;

(b) $M[t] \models \mathscr{T}$ for every t which is random over M;

(c) $M[t] \models$ "Every set is constructible from c and t";

(d) there exists a fixed formula $\Theta(x_1,x_2)$ in $\mathscr{L}'$ (the language of $M[t]$) which defines a well ordering over the elements in ω^ω belonging to $M[t]$ (when $\Theta(x_1,x_2)$ is interpreted in $M[t]$):

(e) for every sentence Ψ in the language of $M[t]$, there exists a Borel set A (in the real world) such that for every random t, $(M[t] \models \Psi) \leftrightarrow t \in A$;

(f) if $\mathscr{T}$ contains a sufficiently large set of axioms, every $\prod_1^1$ relation is absolute.

2.

LEMMA. *Let $B \subset [0,1]$ be a non-countable Borel set. Then there exists a one-to-one Borel function F which takes the unit interval onto B.*

PROOF. The proof is analogous to that of the Cantor-Bernstein theorem. Since B is Borel and non-countable, it includes a perfect set $B_1 \subset B$ [3, p. 447]. Let $Q \subset B_1$ be a countable series of points which is dense in B_1. Then there exists a one-to-one order preserving mapping g of the rational numbers into Q. For all rational $q \in [0,1]$, let $G(q) = g(q)$. For irrational $t \in [0,1]$, let $G(t) = \operatorname{Sup}_{\substack{q \leqq t \\ q \in [0,1]}} g(q)$. G is then a well defined 1 – 1 mapping from $[0,1]$ into B_1. G is clearly Borel (it is in fact arithmetic!).

Let $B_2 = G([0,1])$. B_2 can be described as follows: $b \in B_2 \leftrightarrow$ [(There exists a rational q such that $g(q) = b$) $\vee$ (There exists an increasing series $\{q_i\}$ whose least upper bound is irrational and $g(q_i) < b$ for all i and $\operatorname{Sup}_i g(q_i) = b$)]. Alternately it can be defined as: $b \in B_2 \leftrightarrow$ [(There exists a rational q such that $g(q) = b$) $\vee$ (For all $\varepsilon > 0$ there exists a rational q such that $g(q) < b$ & $|b - g(q)| < \varepsilon$ & for every increasing series $\{q_i\} \subset [0,1]$, if $\operatorname{Sup}_i g(q_i) = b$ then $\operatorname{Sup}_i q_i$ is irrational)]. Thus B_2 is both Σ^1_1 and Π^1_1 and therefore Borel. More generally, the image under G of every Borel subset $C \subset [0,1]$ is Borel. This is shown in the same manner as in the case of B_2. It follows that $G^n C = \underbrace{G \cdot G \cdot \ldots \cdot G(C)}_{n \text{ iterations}}$ is Borel for every Borel subset $C \subset [0,1]$ and every positive integer n. Let A^1 consist of those points in $[0,1]$ not belonging to B. Define F as follows: If $x \in [0,1]$ is such that $x \in G^n A^1$ for some n, let $F(x) = G(x)$; otherwise let $F(x) = x$. F is readily seen to be Borel. Using considerations of the Cantor-Bernstein theorem, we find that F is also one-one and onto.

THEOREM 2. *Let $W \subset [0,1] \times [0,1]$ be a Borel set such that for each $x \in [0,1]$, $W_x \overset{def}{=} \{y \mid (x,y) \in W\}$ is uncountable. Let m be the usual Lebesgue measure on the Borel subsets of $[0,1]$. Then there exists a function $h: [0,1] \times [0,1] \to W$ with the following properties*:

(a) *for each $x \in [0,1]$, the function $h(x,\cdot)$ is one-one and onto W_x and is Borel measurable*;

(b) *for each $y, h(\cdot, y)$ is Lebesgue measurable*; *$h(x,y) \in W_x$ for all x*;

(c) *the function h is Lebesgue measurable.*

PROOF. We combine the approach used in Theorem 1 with the methods of the preceding Lemma.

Let $\xi_0 \in \omega^\omega$ code the Borel set W under the predicate Q_2 described in Theorem 1. Let M be a countable transitive model of ZFC containing ξ_0 such that every set in M is constructible from ξ_0. As was the case in the previous theorem, we need only assume that M fulfills some finite subset of ZFC; however, for the sake of simplicity, we again stipulate that all axioms in ZFC are fulfilled by M. Let $\mathscr{L}$ be a language corresponding to M and let $\mathscr{L}'$ be the language obtained from $\mathscr{L}$ by adding the symbol $\bar{t}$. For each $t \in [0,1]$ which is random over M, let $M[t]$ be the uniquely defined model containing t obtained through the label space procedure·

Let $\Theta(x_1, x_2)$ be a formula in $\mathscr{L}'$ which defines a well ordering over the elements in ω^ω belonging to $M[t]$. In similar fashion, all other symbols defined in the proof of Theorem 1, e.g., $A_2(x_1, x_2)$, $Q_2(x_1, x_2, x_3)$, $B^M_{2,\mu}$ retain the same meanings and usages here.

For random t, let $W_t^{M[t]} \overset{def}{=} \{y \mid (t, y) \in W\} \cap M[t]$. By virtue of the absoluteness of $\prod_1^1$ statements, $W_t^{M[t]} = \{y \mid M[t] \vDash Q_2(\xi_0, t, y)\}$. $W_t^{M[t]}$ is thus definable by interpreting $Q_2(\xi_0, t, y)$ in $M[t]$, and hence $W_t \in M[t]$. $W_t^{M[t]}$ is also Borel in $M[t]$, since it is the cross-section of a Borel set in $M[t]$, namely $B^{M[t]}_{2,\xi_0}$. Let $\eta_t^1 \in M[t]$ be the first element in ω^ω (under the well ordering imposed by Θ) which codes $W_t^{M[t]}$ under the predicate $A_2(x_1, x_2)$. Thus "$\forall y(A_2(\eta_t^1, y) \leftrightarrow Q_2(\xi_0, t, y))$" holds in $M[t]$. By absoluteness, this implies that η_t^1 codes the set $\{y \mid (t, y) \in W\} = W_t$ in the real world. In accordance with Lemma 7 [8, p. 32], the relation "η codes an uncountable Borel set" is absolute. Since $\{y \mid (t, y) \in W\}$ is uncountable in the real world, it therefore follows that $W_t^{M[t]}$ is uncountable in $M[t]$.

It is known that every uncountable Borel set $Y \subset \mathscr{R}$ includes a perfect set as a subset. This is provable within the framework of ZFC and must therefore hold true in $M[t]$. Since perfect sets are Borel, every perfect set has a code. Let $\eta_t^2 \in M[t]$ be the first element (under Θ) in ω^ω such that $M[t] \vDash$ "η_t^2 codes a perfect set, $D_{\eta_t^2}$, which is a subset of $W_t^{M[t]}$." This means that the statement

(6) "the set $\{y \mid Q_2(\eta_t^2, t, y)\}$ is a perfect set and $\forall z[\neg Q_3(\eta_t^2, t, z) \rightarrow Q_2(\eta_t^1, t, z)]$"

holds true in $M[t]$. In accordance with Lemma 7 [8, p. 32], the relation "μ codes a perfect set of reals" is absolute. In addition, the relation $\forall z[\neg Q_3(\eta_t^2, t, z) \rightarrow Q_2(\eta_t^1, t, z)]$ is $\prod_1^1$ and therefore absolute. It follows that (6) is absolute. This, in turn, implies that $D^{\#}{}_{\eta_t^2}$, the set coded by η_t^2 in the real world, is a perfect set and $D^{\#}{}_{\eta_t^2} \subset W_t$. Let $\eta_t^3 \in M[t]$ be the first element in ω^ω that codes a series of points $\{s_i\}$ in $M[t]$ such that $\{s_i\}$ is dense in $D_{\eta_t^2}$. Let $\eta_t^4 \in M[t]$ be the first element in ω^ω that codes a Borel function $g_{\eta_t^4} \in M[t]$ (by means of the graph) defined on the rationals in $[0,1]$ so that for every rational r, $\exists s \in \{s_i\}$ such that $g_{\eta_t^4}(r) = s$ and for all rational r_1, r_2, $r_1 > r_2 \rightarrow g_{\eta_t^4}(r_1) > g_{\eta_t^4}(r_2)$. The function coded by η_t^4 in the real world, $g^{\#}{}_{\eta_t^4}$, will also have the same property. Let $\eta_t^5 \in M[t]$ be the first element in ω^ω that codes the Borel function $G_{\eta_t^5} \in M[t]$ defined by

$$(7) \qquad G_{\eta_t^5}(x) = \begin{cases} g_{\eta_t^4}(x) & \text{for rational } x \in [0,1] \\ \sup\limits_{\substack{r \text{ rational} \\ r < x}} g_{\eta_t^4}(r) & \text{for irrational } x \in [0,1]. \end{cases}$$

It is clear that the defining relation in (7) can be formulated in terms of both $\prod_1^1$ and Σ_1^1 statements. Using the fact that the defining relation in (7) may be expressed in terms of an absolute relation, we may readily show that $G^{\#}{}_{\eta_t^5}$, the Borel set coded by η_t^5 in the real world, constitutes the graph of a 1–1 function defined on $[0,1]$. Moreover, the simultaneous expressibility of $G^{\#}{}_{\eta_t^5}$ in terms of Σ_1^1 and $\prod_1^1$ statements implies that $G^{\#}{}_{\eta_t^5}$ is a Borel function. Let $C_t \in M[t]$ be the set defined by $y \in C_t \leftrightarrow (y \in M[t] \,\&\, \exists x(x \in [0,1] \cap M[t] \,\&\, Q_2(\eta_t^5, x, y))$, i.e., C_t is the image of $[0,1] \cap M[t]$ under $G_{\eta_t^5}$. C_t may alternately be defined by interpreting the following statements in $M[t]$: $y \in C_t \leftrightarrow$ [(There exists a rational q such that $g_{\eta_t^4}(q) = y$) $\vee$ (For all $\varepsilon > 0$ there exists a rational q such that $g_{\eta_t^4}(q) < y$ & $|y - g_{\eta_t}(q)| < \varepsilon$ & for every increasing series of rationals $\{q_i\} \subset [0,1]$, if $\mathrm{Sup}_i\, g_{\eta_t^4}(q_i) = y$ then $\mathrm{Sup}_i\, q_i$ is irrational)]. C_t is thus definable in terms of $\prod_1^1$ and Σ_1^1 statements and is therefore Borel in $M[t]$. Similarly, the image under $G_{\eta_t^5}$ of every Borel set $B \subset [0,1] \cap M[t]$ is likewise Borel (in $M[t]$). Thus for every positive integer n, the function $G^n{}_{\eta_t^5} = \underbrace{G_{\eta_t^5} \cdot G_{\eta_t^5} \cdot \ldots \cdot G_{\eta_t^5}}_{n \text{ iterations}}$ takes $[0,1] \cap M[t]$ onto a Borel set in $M[t]$.

Let $\bar{g}_{\eta_t^4}, \bar{G}_{\eta_t^5}$ be the terms denoting the functions $g_{\eta_t^4}, G_{\eta_t^5}$ in $M[t]$, i.e., $\bar{g}_{\eta_t^4} \equiv$ "$\{(x,y) \mid Q_2(\eta_t^4, x, y)\}$", $\bar{G}_{\eta_t^5} \equiv$ "$\{(x,y) \mid Q_2(\eta_t^5, x, y)\}$". When interpreted in the real world, $\bar{G}_{\eta_t^5}$ denotes the function $G^{\#}{}_{\eta_t^5}$. We assert that for any $\lambda, \mu \in M[t]$ that code Borel subsets of $[0,1]$, the relation

(8) "μ codes the image (under $\bar{G}_{\eta_t^5}$) of the Borel set coded by λ"

is absolute. Relation (8) is true (in $M[t]$ or in the real world) if and only if the relation

(9) $\forall y[(A_2(\mu, y)$ & [(There exists a rational q such that $A_2(\lambda, q)$ & $\bar{g}_{\eta_t^4}(q) = y$) $\vee$ (For all $\varepsilon > 0$ there exists a rational q such that $\bar{g}_{\eta_t^4}(q) < y$ & $|y - \bar{g}_{\eta_t^4}(q)| < \varepsilon$ & for every increasing series of rationals $\{q_i\} \subset [0,1]$, if $\mathrm{Sup}_i\, |\bar{g}_{\eta_t^4}(q_i) = y$ then $\mathrm{Sup}_i\, q_i$ is irrational and $A_2(\lambda, \mathrm{Sup}_i\, q_i)$)]) $\vee$ $(A_3(\mu, y)$ & $\forall x(\neg A_3(\lambda, x) \rightarrow Q_3(\eta_t^5, x, y))]$

holds true. However, (9) is equivalent to a $\prod_1^1$ statement and is therefore absolute.

Let $\eta_t^6 \in M[t]$ be the first element in ω^ω which codes the set $A_{\eta_t^6} = \{x \mid x \in M[t]$ & $x \in [0,1]$ & $x \notin W_t^{M[t]}\}$. Then $A^{\#}{}_{\eta_t^6}$, the set coded by η_t^6 in the real world, is equal to $[0,1] - W_t$. For every natural n, the image of $A_{\eta_t^6}$ under $G^n{}_{\eta_t^5}$, $G^n{}_{\eta_t^5}(A_{\eta_t^6})$, is Borel in $M[t]$. Let $\eta_t^7 \in M[t]$ be the first element in ω^ω which codes the Borel set $E_{\eta_t^7} \overset{def}{=} \cup_n G^n{}_{\eta_t^5}(A_{\eta_t^6})$ in $M[t]$. Then $E^{\#}{}_{\eta_t^7} = \cup_n G^{\#}{}_{\eta_t^5}(A^{\#}{}_{\eta_t^6})$ as follows easily from the absoluteness of (8). Let $\eta_t^8 \in M[t]$ be the first element in ω^ω such that η_t^8 codes the Borel function defined by

$$(10) \qquad F_{\eta_t^8}(x) \overset{def}{=} \begin{cases} G_{\eta_t^5}(x) & x \in [0,1] \,\&\, x \in E_{\eta_t^7} \\ x & x \in [0,1] \,\&\, x \notin E_{\eta_t^7}. \end{cases}$$

The defining condition in (10) is equivalent to the statement

$$(11) \qquad \forall x \forall y (Q_2(\eta_t^8, x, y) \leftrightarrow [(x \in [0,1] \,\&\, A_2(\eta_t^7, x) \,\&\, Q_2(\eta_t^5, x, y)) \vee (x \in [0,1] \,\&\, A_3(\eta_t^7, x) \,\&\, x = y)])$$

when interpreted in $M[t]$. However, (11) is equivalent to a $\prod_1^1$ statement and is therefore absolute. Interpreting (11) in the real world, we obtain that $F^{\#}_{\eta_t^8}$, the element coded by η_t^8 in the real world, is a 1-1 function defined on $[0,1]$ such that

$$F^{\#}_{\eta_t^8}(x) = \begin{cases} G^{\#}_{\eta_t^5}(x) & x \in [0,1] \,\&\, x \in E^{\#}_{\eta_t^7} \\ x & x \in [0,1] \,\&\, x \notin E^{\#}_{\eta_t^7}. \end{cases}$$

Using considerations of the Cantor-Bernstein theorem as in the Lemma, we find that $F^{\#}_{\eta_t^8}$ takes the unit interval one-to-one onto W_t.

Let

$$u(t) = \begin{cases} \eta_t^8 & \text{for random } t \in [0,1] \\ 0 & \text{for non-random } t \in [0,1]. \end{cases}$$

We assert that $u(t)$ is Borel.

For any rational r_1, r_2, $r_1 \leqq r_2$, let $P_{r_1,r_2} = \{t \mid u(t) \in [r_1, r_2]\}$. $P_{r_1,r_2} = \{t \mid t$ is random & $\eta_t^8 \in [r_1, r_2]\} \cup \{t \mid t$ is non-random & $r_1 \leqq 0 \leqq r_2\}$. Since the set of all random t's is Borel, the set $\{t \mid t$ is non-random & $r_1 \leqq 0 \leqq r_2\}$ is likewise Borel. $\{t \mid t$ is random & $\eta_t^8 \in [r_1, r_2]\} = \{t \mid t$ is random & $M[t] \models$ "$\eta_t^8 \in [r_1, r_2]$"$\}$. However, the relation "$\eta_t^8 \in [r_1, r_2]$" is expressible in $\mathscr{L}'$. Thus by Theorem II, 2.8 [8, p. 38], the set $\{t \mid t$ is random & $M[t] \models$ "$\eta_t^8 \in [r_1, r_2]$"$\}$ is Borel. Hence P_{r_1,r_2} is Borel. Since P_{r_1,r_2} is the inverse image of an arbitrarily selected interval, it follows that $u(t)$ is Borel.

In accordance with the Lemma and the axiom of choice, there exists a function $\mathscr{H}(x): [0,1] \to \omega^\omega$ such that for all $x \in [0,1]$, $\mathscr{H}(x)$ codes a $1-1$ Borel function from $[0,1]$ onto W_x. Let

$$\hat{u}(t) \overset{def}{=} \begin{cases} \eta_t^8 & \text{for random } t \in [0,1] \\ \mathscr{H}(t) & \text{for non-random } t \in [0,1]. \end{cases}$$

We are now in a position to define h:

$h(t,y) \overset{def}{=}$ the unique $z \in [0,1]$ such that $Q_2(\hat{u}(t), y, z)$. We must show, however, that h fulfills a)–c) as described in the formulation of the theorem.

a) The fact that for non-random t, $h(t,\cdot)$ is 1-1, onto W_t and Borel follows immediately from the definition of h; for random t this derives from the absoluteness of the properties fulfilled by η_t^8.

b) For any fixed $y_1 \in [0,1]$ and any rational interval $[r_1, r_2]$, $\{t \mid h(t,y_1) \in [r_1,r_2]\} = \{t \mid h(t,y_1) \in [r_1,r_2]$ and t is non-random$\} \cup \{t \mid \forall z(Q_2(\eta_t^8, y_1, z) \rightarrow z \in [r_1,r_2])$ & t is random$\}$. Let λ_1 code the Borel set of random t's and let λ_2 code the Borel function $u(t)$. Then

$$\begin{aligned} &\{t \mid \forall z(Q_2(\eta_t^8, y_1, z) \rightarrow z \in [r_1,r_2]) \ \&\ t \text{ is random}\} \\ &= \{t \mid \forall z \forall w((\neg Q_3(\lambda_2, t, w) \ \&\ \neg Q_3(w, y_1, z)) \rightarrow z \in [r_1,r_2]) \ \&\ A_2(\lambda_1, t)\} \\ &= \{t \mid \exists z \exists w(\neg Q_3(\lambda_2, t, w) \ \&\ \neg Q_3(w, y_1, z) \ \&\ z \in [r_1,r_2] \ \&\ \neg A_3(\lambda_1, t)\}. \end{aligned}$$

This is simultaneously Σ_1^1 and $\prod_1^1$ and is therefore Borel. On the other hand $\{t \mid h(t,y_1) \in [r_1,r_2]$ & t is non-random$\}$ is of Lebesgue measure zero. Thus $\{t \mid h(t,y_1) \in [r_1,r_2]\}$ is Lebesgue measurable for every rational interval.

c) is proven in much the same way as b).

Q. E. D.

3.

To the best of my knowledge, all known proofs of the fact that the power axiom holds in $M[t]$ make use of the replacement axiom of ZF. The weaker set of axioms in Zermelo set theory (in particular the separation axiom) appears to be insufficient to establish this result. At the same time, the proofs in this article seem to depend upon the validity of the power axiom in $M[t]$. Another proposition which appears essential to our proofs is the existence of a transitive model M of a sufficiently large (but finite) subset $\mathscr{T}$ of the axioms of ZFC. $\mathscr{T}$ must be rich enough to guarantee that all Σ_1^1 and $\prod_1^1$ relations that hold in M hold true in the real world as well. The existence of such models in all likelihood cannot be proven by means of the Zermelo axioms alone. Thus, although the theorems presented here and in [10] are analytical in nature, the provability of some of them may depend upon the stronger axioms in ZFC.† It would be interesting to know whether these theorems can be proven if one uses only the axioms of Zermelo set theory.

† A stronger version of Theorem 1 was proven by N. Lusin in [5, p. 244].

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